Journal of Engineering Physics and Thermophysics, Vol. 69, No. 5, 1996

HEAT CONDUCTION OF ORTHOTROPIC HALF-SPACE FOR MIXED DISCONTINUOUS BOUNDARY CONDITIONS

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UDC 517.966

The nonstationary temperature field in a half-space heated across a circular region with a known radius $r = R$ over a surface $z = 0$ of a semibounded body is found. Outside the circular region $r > R$ the initial temperature $\theta(r, 0, \tau) = T(r, 0, \tau) - T_0 = 0$ is maintained on the surface $z = 0$. Particular regularities of development of nonstationary temperature fields along the axis $r = 0$, $z \geq 0$ and over the surface $z = 0$, $0 \leq r < R$ are given.

The problem of increasing the accuracy of thermophysical measurements is inseparably linked with the creation of precision tools for implementation of the theoretically postulated boundary conditions that are the basis of any method of determination of thermophysical characteristictics. In a technique for a thermophysical experiment, two kinds of boundary conditions can be realized most simply and sufficiently: constancy of the surface temperature of the investigated body and constancy of the specific heat flux if the latter is initiated by a quick-response electric heater.

The present work is aimed at solving the relevant original two-dimensional problem on transient heat contuction for isotropic and orthotropic half-spaces when the indicated boundary conditions on the surface ($z = 0$) of the investigated semibounded body exist simultaneously, i.e., in the region ($z = 0$, $0 \leq r \leq R$) a constant specific heat flux q_0 = const is prescribed, while outside the circle ($z = 0$, $r > R$) a constant temperature equal to the initial T_0 is maintained.

Solution of such two-dimensional problems of mathematical physics in the presence of mixed discontinuous boundary conditions of the first and second kind usually involves the analysis (solution) of paired integral equations [1, 2].

In the present work it is shown that an analytical solution of the formulated two-dimensional problem of transient heat conduction with mixed discontinuous boundary conditions for certain ranges of cylindrical coordinates (r, z) in the investigated body can be constructed without solving directly the paired integral equations.

Mathematical Formulation of the Problem. It is required to find a solution of the following system of differential equations for the functions

$$\theta_1 (r, z, \tau) = T_1 (r, z, \tau) - T_0 = \theta_1 \quad (0 \leq r \leq R\,,\ \ z > 0\,,\ \ \tau > 0)$$

and

$$\theta_2 (r, z, \tau) = T_2 (r, z, \tau) - T_0 = \theta_2 \quad (r > R\,,\ \ z > 0\,,\ \ \tau > 0):$$

$$\frac{\partial^2 \theta_1}{\partial r^2} + \frac{1}{r} \frac{\partial \theta_1}{\partial r} + \frac{a_z}{a_r} \frac{\partial^2 \theta_1}{\partial z^2} = \frac{1}{a_r} \frac{\partial \theta_1}{\partial \tau}, \quad 0 \leq r < R; \tag{1}$$

$$\frac{\partial^2 \theta_2}{\partial r^2} + \frac{1}{r} \frac{\partial \theta_2}{\partial r} + \frac{a_z}{a_r} \frac{\partial^2 \theta_2}{\partial z^2} = \frac{1}{a_r} \frac{\partial \theta_2}{\partial \tau}, \quad R < r < \infty, \tag{2}$$

Belarusian State University, Minsk, Belarus. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 69, No. 5, pp. 826-833, September-October, 1996. Original article submitted May 21, 1996.

1062-0125/96/6905-0643$15.00

at the boundary conditions

$$\theta_1 (r, z, 0) = \theta_2 (r, z, 0) = 0 , \tag{3}$$

$$-\frac{\partial \theta_1 (r, 0, \tau)}{\partial z} = \frac{q_0}{\lambda_z}, \quad |r| < R , \quad z = 0 , \quad \tau > 0 , \tag{4}$$

$$\theta_2 (r, 0, \tau) = 0 , \quad |r| > R , \quad z = 0 , \quad \tau \geq 0 , \tag{5}$$

$$\frac{\partial \theta_1 (0, z, \tau)}{\partial r} = 0 , \quad r = 0 , \quad z > 0 , \quad \tau \geq 0 , \tag{6}$$

$$\frac{\partial \theta_1 (r, \infty, \tau)}{\partial z} = \frac{\partial \theta_2 (r, \infty, \tau)}{\partial z} = \frac{\partial \theta_2 (\infty, z, \tau)}{\partial r} = 0 , \tag{7}$$

$$\theta_1 (R, z, \tau) = \theta_2 (R, z, 0) , \quad z > 0 , \tag{8}$$

$$\frac{\partial \theta_1 (R, z, \tau)}{\partial r} = \frac{\partial \theta_2 (R, z, \tau)}{\partial r} , \quad z > 0 . \tag{9}$$

Sequence of the Solution. *Step 1.* At $\partial \theta_i / \partial \tau = 0$, $i = 1, 2$, the solution of the steady-state two-dimensional problem for the function $\theta_1 (r, z) = \theta_2 (r, z)$ (the body is isotropic, $K_a = 1$, where $K_a = a_r / a_z$ is known [4]) is

$$\theta (r, z) = \frac{2 q_0}{\lambda \pi} \int_0^\infty \exp (- xz) J_0 (xr) \frac{\sin xR - xR \cos xR}{x^2} dx . \tag{10}$$

Step 2. Now we write the Fourier sine and cosine transformations of solution (10) for $|r| > R$ and $|r| < R$:

$$\theta_c (r, p) = \int_0^\infty \theta (r, z) \cos pz dz = \frac{2 q_0}{\lambda \pi} \left\{ \frac{\pi}{2} I_0 (pr) \int_r^R t \exp (- tp) \, dt + \right.$$

$$\left. + \frac{\pi}{2} I_0 (pr) \int_0^r t \exp (- tp) \, dt - \frac{\pi}{2p^2} I_0 (pr) + \frac{\pi}{2p^2} \right\} =$$

$$= \frac{q_0}{\lambda} \left\{ I_0 (pr) \int_0^R t \exp (- tp) \, dt - \frac{1}{p^2} I_0 (pr) + \frac{1}{p^2} \right\} , \quad |r| < R ; \tag{11}$$

$$\theta_s (r, p) = \int_0^\infty \theta (r, z) \sin pz dz = \frac{2 q_0 p}{\lambda \pi} \int_0^R \frac{t}{p} \operatorname{sh} (tp) K_0 (pr) \, dt =$$

$$= \frac{2 q_0}{\lambda \pi} K_0 (pr) \int_0^R t \operatorname{sh} (tp) \, dt , \quad |r| > R . \tag{12}$$

Summary of step 2:

$$\frac{1}{p^2} - I_0(pr)\frac{\exp(-Rp)(1+pR)}{p^2} = \frac{2}{\pi}\int_0^{\infty} \frac{xJ_0(xr)}{x^2+p^2}\,\frac{\sin xR - xR\cos xR}{x^2}\,dx\,, \tag{13}$$

$$r \le R\,;$$

$$K_0(pr)\frac{pR\,\mathrm{ch}\,pR - \mathrm{sh}\,pR}{p^2} = p\int_0^{\infty} \frac{J_0(xr)}{x^2+p^2}\,\frac{\sin xR - xR\cos xR}{x^2}\,dx\,, \tag{14}$$

$$r \ge R\,.$$

Step 3. Applying Laplace and, respectively, Fourier cosine and sine transformations to differential equations of transient heat conduction (1) and (2), it is easy to obtain a solution for $\bar{\theta}_{1c}(r, p, s)$ and $\bar{\theta}_{2c}(r, p, s)$ in the form ($K_a = 1$):

$$\bar{\theta}_{1c}(r, p, s) = \frac{\bar{q}(s)}{\lambda\left(p^2 + \frac{s}{a}\right)} + A(p, s)\, I_0\left(r\sqrt{\left(p^2 + \frac{s}{a}\right)}\right), \quad r < R\,; \tag{15}$$

$$\bar{\theta}_{2c}(r, p, s) = B(p, s)\, K_0\left(r\sqrt{\left(p^2 + \frac{s}{a}\right)}\right), \quad r > R\,. \tag{16}$$

Using boundary conditions (7) and (8), we arrive at the following system of equations:

$$\int_0^{\infty}\left\{\frac{\bar{q}(s)}{\lambda\left(p^2 + \frac{s}{a}\right)} + A(p, s)\, I_0\left(R\sqrt{\left(p^2 + \frac{s}{a}\right)}\right)\right\}\cos pz\,dp =$$

$$= \int_0^{\infty} B(p, s)\, K_0\left(R\sqrt{\left(p^2 + \frac{s}{a}\right)}\right)\sin pz\,dp\,, \tag{17}$$

$$\int_0^{\infty} A(p, s)\sqrt{\left(p^2 + \frac{s}{a}\right)}\; I_1\left(R\sqrt{\left(p^2 + \frac{s}{a}\right)}\right)\cos pz\,dp =$$

$$= -\int_0^{\infty} B(p, s)\sqrt{\left(p^2 + \frac{s}{a}\right)}\; K_1\left(R\sqrt{\left(p^2 + \frac{s}{a}\right)}\right)\sin pz\,dp\,. \tag{18}$$

Step 4. We now use an integral relationship between the Fourier sine and cosine transformations of the following form

$$\int_0^{\infty} \mu(s, p)\sin pz\,dp = \int_0^{\infty} g(s, p)\cos pz\,dp\,, \tag{19}$$

where

$$\mu(s, p) = p\int_0^{\infty} \frac{f(x)\,dx}{\left(x^2 + p^2 + \frac{s}{a}\right)\sqrt{\left(p^2 + \frac{s}{a}\right)}}\,, \tag{20}$$

$$g(s,p)=\int_0^\infty \frac{f(x)\,dx}{x^2+p^2+\frac{s}{a}}. \tag{21}$$

Applying relations (19)-(21) to each of Eqs. (17), (18), we can write

$$g_1(s,p)=\frac{\bar q(s)}{\lambda\left(p^2+\frac{s}{a}\right)}+A(p,s)\,I_0\left(R\sqrt{\left(p^2+\frac{s}{a}\right)}\right)=\int_0^\infty \frac{f_1(x)\,dx}{x^2+p^2+\frac{s}{a}}, \tag{22}$$

$$\mu_1(s,p)=B(p,s)\,K_0\left(R\sqrt{\left(p^2+\frac{s}{a}\right)}\right)=p\int_0^\infty \frac{f_1(x)\,dx}{\left(x^2+p^2+\frac{s}{a}\right)\sqrt{\left(p^2+\frac{s}{a}\right)}}, \tag{23}$$

$$g_2(s,p)=A(p,s)\sqrt{\left(p^2+\frac{s}{a}\right)}\,I_1\left(R\sqrt{\left(p^2+\frac{s}{a}\right)}\right)=$$

$$=\int_0^\infty \frac{f_2(x)\,dx}{\left(x^2+p^2+\frac{s}{a}\right)\sqrt{\left(p^2+\frac{s}{a}\right)}}, \tag{24}$$

$$\mu_2(s,p)=-B(p,s)\sqrt{\left(p^2+\frac{s}{a}\right)}\,K_1\left(R\sqrt{\left(p^2+\frac{s}{a}\right)}\right)=$$

$$=p\int_0^\infty \frac{f_2(x)\,dx}{\left(x^2+p^2+\frac{s}{a}\right)\sqrt{\left(x^2+\frac{s}{a}\right)}}. \tag{25}$$

Let us pass to Eq. (13) in which the complex $\sqrt{p^2+s/a}$ and $r=R$ can be taken as the parameter p. Then

$$\frac{1}{p^2+\frac{s}{a}}-I_0\left(R\sqrt{\left(p^2+\frac{s}{a}\right)}\right)\frac{\exp\left(-R\sqrt{\left(p^2+\frac{s}{a}\right)}\right)\left(1+R\sqrt{\left(p^2+\frac{s}{a}\right)}\right)}{p^2+\frac{s}{a}}=$$

$$=\frac{2}{\pi}\int_0^\infty \frac{xJ_0(xR)}{x^2+p^2+\frac{s}{a}}\,\frac{\sin xR-xR\cos xR}{x^2}\,dx. \tag{26}$$

Multiplying (26) by $\bar q(s)/\lambda$ and comparing the obtained expression with Eq. (22), we find that

$$f_1(x)=\frac{2}{\pi}J_0(xR)\frac{\sin xR-xR\cos xR}{x}\,\frac{\bar q(x)}{\lambda}, \tag{27}$$

$$A(p,s)=-\frac{\bar q(s)}{\lambda}\,\frac{\exp\left(-R\sqrt{\left(p^2+\frac{s}{a}\right)}\right)\left(1+R\sqrt{\left(p^2+\frac{s}{a}\right)}\right)}{p^2+\frac{s}{a}}. \tag{28}$$

According to (23) and (27) we have

$$B(p,s)\,K_0\left(R\sqrt{\left(p^2+\frac{s}{a}\right)}\right)=\frac{2p\bar{q}(s)}{\pi\lambda}\int_0^\infty\frac{J_0(xR)}{\left(x^2+p^2+\frac{s}{a}\right)\sqrt{\left(x^2+\frac{s}{a}\right)}}\times$$

$$\times\frac{\sin xR-xR\cos xR}{x}\,dx\,. \tag{29}$$

If we differentiate Eq. (13) with respect to r and assume that $p\to\sqrt{p^2+s/a}$, then employing expressions (24) and (25) it is easy to find that at $r=R$

$$f_2(x)=-\frac{2\bar{q}(s)}{\pi\lambda}J_1(xR)\,(\sin xR-xR\cos xR)\,, \tag{30}$$

$$B(p,s)\sqrt{\left(p^2+\frac{s}{a}\right)}\,K_1\left(R\sqrt{\left(p^2+\frac{s}{a}\right)}\right)=-\frac{2p\bar{q}(s)}{\pi\lambda}\int_0^\infty\frac{J_1(xR)}{x^2+p^2+\frac{s}{a}}\times$$

$$\times\frac{\sin xR-xR\cos xR}{\sqrt{\left(x^2+\frac{s}{a}\right)}}\,dx\,. \tag{31}$$

Thus, solutions (15) and (16) with the use of (28), (29), and (31) can be written as:

$$\theta_{1c}(r,p,s)\Big|_{r<R}=\frac{\bar{q}(s)}{\lambda\left(p^2+\frac{s}{a}\right)}=\frac{\bar{q}(s)}{\lambda}I_0\left(r\sqrt{\left(p^2+\frac{s}{a}\right)}\right)\times$$

$$\times\frac{\exp\left(-R\sqrt{\left(p^2+\frac{s}{a}\right)}\right)\left(1+R\sqrt{\left(p^2+\frac{s}{a}\right)}\right)}{p^2+\frac{s}{a}}\,. \tag{32}$$

$$\theta_{2s}(r,p,s)\Big|_{r>R}=\frac{2\bar{q}(s)}{\pi\lambda}\,\frac{K_0\left(r\sqrt{\left(p^2+\frac{s}{a}\right)}\right)}{K_0\left(R\sqrt{\left(p^2+\frac{s}{a}\right)}\right)}\,p\int_0^\infty\frac{J_0(xR)}{x^2+p^2+\frac{s}{a}}\times$$

$$\times\frac{\sin xR-xR\cos xR}{\sqrt{\left(x^2+\frac{s}{a}\right)}}\,dx=-\frac{2\bar{q}(s)}{\pi\lambda}\,\frac{K_0\left(r\sqrt{\left(p^2+\frac{s}{a}\right)}\right)}{\sqrt{\left(p^2+\frac{s}{a}\right)}\,K_1\left(R\sqrt{\left(p^2+\frac{s}{a}\right)}\right)}\times$$

$$\times p\int_0^\infty\frac{J_1(xR)\,(\sin xR-xR\cos xR)}{\left(x^2+p^2+\frac{s}{a}\right)\sqrt{\left(x^2+\frac{s}{a}\right)}}\,dx\,,\quad -\frac{\pi}{2}<\arg\sqrt{s}\le\frac{\pi}{2}\,. \tag{33}$$

Since for further studies and practical applications it is of interest to have a solution for the temperature field $\theta_1(r,z,\tau)$ in the first region $(0\le r<R,\ z\ge 0,\ \tau>0)$, we find the inverse transform of (32):

$$\bar{\theta}_1(r,z,s) = \frac{\bar{q}(s)\sqrt{a}}{\lambda\sqrt{s}}\exp\left(-z\frac{\sqrt{s}}{\sqrt{a}}\right) - \frac{2\bar{q}(s)}{\pi\lambda}\int_0^\infty \frac{I_0\left(r\sqrt{\left(p^2+\frac{s}{a}\right)}\right)}{p^2+\frac{s}{a}} \times$$

$$\times \exp\left(-R\sqrt{\left(p^2+\frac{s}{a}\right)}\right)\cos pzdp - \frac{2\bar{q}(s)R}{\pi\lambda}\int_0^\infty I_0\left(r\sqrt{\left(p^2+\frac{s}{a}\right)}\right) \times$$

$$\times \frac{\exp\left(-R\sqrt{\left(p^2+\frac{s}{a}\right)}\right)}{\sqrt{\left(p^2+\frac{s}{a}\right)}}\cos pzdp = \frac{2\bar{q}(s)\sqrt{a}}{\pi\lambda} \times$$

$$\times \int_0^\infty \frac{\exp\left(-\frac{z}{\sqrt{a}}\sqrt{x^2+s}\right)}{\sqrt{x^2+s}} J_0\left(\frac{xr}{\sqrt{a}}\right)\sin\left(\frac{Rx}{\sqrt{a}}\right)\frac{dx}{x} - \frac{2\bar{q}(s)R}{\pi\lambda} \times$$

$$\times \int_0^\infty \frac{\exp\left(-\frac{z}{\sqrt{a}}\sqrt{x^2+s}\right)}{\sqrt{x^2+s}} J_0\left(\frac{xr}{\sqrt{a}}\right)\cos\left(\frac{Rx}{\sqrt{a}}\right)dx =$$

$$= \frac{2\bar{q}(s)}{\pi\lambda}\int_0^\infty \frac{\exp\left(-z\sqrt{\left(x^2+\frac{s}{a}\right)}\right)}{\sqrt{\left(x^2+\frac{s}{a}\right)}} J_0(xr)\frac{\sin(xR) - xR\cos(xR)}{x}dx, \tag{34}$$

or at $\bar{q}(s) = q_0/s$ in the time domain we have

$$\theta_1(r,z,\tau) = \frac{q_0 R}{\pi\lambda}\int_0^\infty J_0\left(\frac{r}{R}x\right)\frac{\sin x - x\cos x}{x^2} \times$$

$$\times \left\{\exp\left(-\frac{z}{R}x\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{a\tau}} - \frac{x}{R}\sqrt{a\tau}\right) - \right.$$

$$\left. - \exp\left(-\frac{z}{R}x\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{a\tau}} + \frac{x}{R}\sqrt{a\tau}\right)\right\}dx. \tag{35}$$

For an orthotropic body, solutions of (34) and (35) are easy to obtain in the following form:

$$\bar{\theta}_1(r,z,s) = \frac{2\bar{q}(s)\sqrt{a_z}}{\pi\lambda_z}\int_0^\infty \frac{\exp\left(-\frac{z}{\sqrt{a_z}}\sqrt{s+a_r x^2/R^2}\right)}{\sqrt{s+a_r x^2/R^2}}$$

$$J_0\left(\frac{r}{R}x\right)\frac{\sin x - x\cos x}{x^2}dx, \quad 0 \le r < R; \tag{36}$$

$$\theta_1(r,z,\tau)\Big|_{\substack{q(\tau)=q_0\\ r<R}} = \frac{q_0 R}{\pi\lambda_z\sqrt{K_a}}\int_0^\infty J_0\left(\frac{r}{R}x\right)\frac{\sin x - x\cos x}{x^2}\times$$

$$\times\left\{\exp\left(-\frac{z}{R}\sqrt{K_a}\,x\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{a_z\tau}}-\frac{x}{R}\sqrt{a_r\tau}\right)-\right.$$

$$\left.-\exp\left(\frac{z}{R}\sqrt{K_a}\,x\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{a_z\tau}}+\frac{x}{R}\sqrt{a_r\tau}\right)\right\}dx\,. \tag{37}$$

On the axis $r = 0$ $(z, \tau \geq 0)$ solution (37) acquires the form

$$\theta_1(0,z,\tau) = \frac{2q_0\sqrt{\tau}}{b_z\sqrt{\pi}}\left\{\exp\left(-\frac{z^2}{4a_z\tau}\right)\operatorname{erf}\left(\frac{R}{2\sqrt{a_r\tau}}\right)-\frac{z}{\sqrt{\pi a_z\tau}}\times\right.$$

$$\left.\times\left[\arctan\left(\frac{R}{z\sqrt{K_a}}\right)-\frac{\sqrt{\pi}}{2}\int_0^{\frac{z}{2\sqrt{a_z\tau}}}\operatorname{erf}\left(\frac{Rx}{z\sqrt{K_a}}\right)\exp(-x^2)\,dx\right]\right\}. \tag{38}$$

Recall that $b_z = \lambda_z/\sqrt{a_z}$, $K_a = a_r/a_z$. The excess temperature at the center of a heating spot at $r = z = 0$ on the surface of the considered orthotropic body is determined by the following expression:

$$\theta_1(0,0,\tau) = \frac{2q_0\sqrt{\tau}}{b_z\sqrt{\pi}}\operatorname{erf}\left(\frac{R}{2\sqrt{a_r\tau}}\right). \tag{39}$$

In a steady-state thermal regime $(\tau \to \infty)$ solution (37) has an analytical extension to the second region $r > R$. Thus, for any point r and z we have

$$\theta_i(r,z,\infty) = \frac{2q_0R}{b_z\lambda\sqrt{\pi}}\int_0^\infty \exp\left(-\frac{z}{R}\sqrt{K_a}x\right)J_0\left(\frac{r}{R}x\right)\times$$

$$\times\frac{\sin x - x\cos x}{x^2}\,dx\,,\quad i = 1, 2\,. \tag{40}$$

At $K_a = 1$ we have the corresponding steady-state solution from (40) for an isotropic half-space heated across a circular region $(0 \leq r < R,\ z = 0)$ by a constant heat flux with temperature T_0 outside the circle $(r > R,\ z = 0)$ being maintained constant at the boundary of the given body [4].

At $z = 0$ from (40) the value (distribution) of the steady-state temperature on the surface of the orthotropic half-space is obtained in the region of a circular heating spot $(0 \leq r < R)$:

$$\theta_i(r,0,\infty) = \begin{cases}\dfrac{2q_0R}{\pi\lambda_z\sqrt{K_a}}\sqrt{\left(1-\dfrac{r^2}{R^2}\right)}, & r < R\,;\\[2ex] 0\,, & r \geq R\,.\end{cases} \tag{41}$$

The stationary value of excess temperature (4) on the axis $r = 0$ $(z \geq 0)$ can be represented as

$$\theta_1(0, z, \infty) = \frac{2q_0R}{\pi\lambda_z\sqrt{K_a}}\left\{1 - \frac{z}{R}\sqrt{K_a}\arctan\left(\frac{R}{z\sqrt{K_a}}\right)\right\}, \tag{42}$$

and according to (41) and (42) at the central point of the circular heating spot it acquires the simple form

$$\theta(0, 0, \infty) = \frac{2q_0R}{\pi\lambda_z\sqrt{K_a}}. \tag{43}$$

The relationship between the specific heat flux $\lambda_z\partial_t\theta(r, 0, \infty)/\partial z = q^*(r)$ at any point of the boundary surface $z = 0$, $r \geq 0$ and the prescribed specific heat flux $q_0 = \text{const}$ is as follows

$$\left.\frac{q^*(r)}{q_0}\right|_{z=0} = \begin{cases} -1, & r \leq R, \\ \frac{2}{\pi}\left[\frac{1}{\sqrt{\left(\frac{r^2}{R^2} - 1\right)}} - \arcsin\left(\frac{R}{r}\right)\right], & r > R. \end{cases}$$

Based on the above analytical representations, formulas can be obtained for determination of the thermophysical properties of orthotropic materials (without loss of their integrity) provided that theoretically postulated boundary conditions (3)-(9) are realized in the technique of the thermophysical experiment.

NOTATION

$\theta_1(r, z, \tau)$, $\theta_2(r, z, \tau)$, excess temperatures in the corresponding domains of the variable r (throughout the text); r_0, r, R, radius of circle and cylindrical coordinates, respectively; $q(\tau)$, q_0, specific heat flux; $K_a = a_r/a_z$, parameter characterizing the relationships between thethermophysical properties in the corresponding directions; s, p, parameters of the integral Laplace and Hankel transformations; $J_0(x)$, $J_1(x)$, Bessel functions of the zeroth and first order; $I_0(x)$, $I_1(x)$, $K_0(x)$, $K_1(x)$, modified Bessel functions of the corresponding order.

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